J. Math. Anal. Appl. 379 (2011) 788–801

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Continuity properties of solutions to some degenerate elliptic equations ☆

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ARTICLE INFO

Article history:
Received 30 July 2010
Submitted by V. Radulescu

Keywords:
Hölder
Lipschitz
Nonlinear
Degenerate
Elliptic
Pde
Convex
Nonsmooth

ABSTRACT

We consider a nonlinear (possibly) degenerate elliptic operator $Lv = -\operatorname{div} a(\nabla v) + b(x, v)$ where the field a and the function b are (unnecessarily strictly) monotonic and a satisfies a very mild ellipticity assumption. For a given boundary datum ϕ we prove the existence of the maximum and the minimum of the solutions and formulate a Haar–Radò type result, namely a continuity property for these solutions that may follow from the continuity of ϕ. In the homogeneous case we formulate some generalizations of the Bounded Slope Condition and use them to obtain the Lipschitz or local Lipschitz regularity of solutions to $Lu = 0$. We prove the global Hölder regularity of the solutions in the case where ϕ is Lipschitz.

1. Introduction

A famous result due to Hartman and Stampacchia in 1966 [1] shows the existence of a globally Lipschitz solution to the equation

$$-\operatorname{div} a(\nabla v) + F(u) = 0 \quad \text{on } \Omega, \qquad u = \phi \quad \text{on } \partial\Omega \tag{1.1}$$

when the boundary datum ϕ satisfies the *Bounded Slope Condition* (*BSC*); we refer to Section 5 for further details. It is required moreover that $a : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and satisfies the uniform ellipticity condition

$$\forall \xi, \eta \in \mathbb{R}^n \quad \big(a(\xi) - a(\eta)\big) \cdot (\xi - \eta) \geqslant \mu |\xi - \eta|^2 \quad (\mu > 0) \tag{1.2}$$

and F fulfills some technical assumptions that we omit here. In the case where $F = 0$ the result is obtained even under the weaker assumption that a is just (unnecessarily strictly) monotonic. These results follow the method presented by Stampacchia in [2] where the author proves, under the same assumptions on the boundary datum, the existence of a minimizer of an integral functional among Lipschitz functions. The basic tool used by Stampacchia is the construction of barriers without making use of growth assumptions or ellipticity conditions, apart the strict convexity of the Lagrangian; one of the aims of [2] is to consider problems with slow growth as the minimal surface one. After many years this result became

☆ Partially supported by the "Progetto di ateneo dell'Università degli Studi di Padova" No. CPDA081458/08 and PRIN "Equazioni e sistemi di tipo ellittico: stime a priori, esistenza, regolarità" (2008).

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2011.02.001

a source of inspiration for some new results in the Calculus of Variations concerning the regularity of the minimizers for the problem

$$\min \int_{\Omega} f\big(\nabla v(x)\big)\,dx: \quad u \in \phi + W_0^{1,1}(\Omega). \tag{P}$$

We mention Cellina who first revisited their paper in this framework and established in [3] the *Lipschitz continuity* of the minimizers of (P) when ϕ satisfies the (BSC); Clarke in [4] introduced the new one sided *Lower/Upper* (BSC) and obtained under this condition the *local Lipschitz* continuity of the solutions to (P) by assuming moreover that Ω is convex. In both cases the Lagrangian f was supposed to be strictly convex due mostly to the lack of the validity of the Comparison Principles when the epigraph of f has some non-trivial flat faces. The methods developed by these authors allowed Bousquet to prove in [5] the continuity of the minimizers for a continuous boundary datum, and us to establish in [6] the *global Hölder continuity* of the minimizers of (P) once ϕ is Lipschitz and f is coercive; there we were also able to drop the usual strict convexity assumption on the Lagrangian.

Following ideally the same path of Stampacchia these latter results obtained in the framework of the Calculus of Variations are now giving some new existence theorems in the framework of the Pde's. Bousquet considered the very same operator studied in [1] and obtained in [7] the existence of a solution to (1.1) among locally Lipschitz functions if ϕ satisfies Clarke's unilateral (BSC) and a is uniformly elliptic.

Our purpose is to study the existence of regular solutions to (1.1) when the field a is not uniformly elliptic. More precisely we are concerned with the problem

$$Lv := -\operatorname{div} a(\nabla v) + b(x, v) = 0 \quad \text{on } \Omega \subset \mathbb{R}^n, \qquad u = \phi \quad \text{on } \partial\Omega$$

where the field a, different from [7], is *not* supposed to be uniformly elliptic, but just to satisfy

$$\forall \xi, \eta \in \mathbb{R}^n \quad \big(a(\eta) - a(\xi)\big) \cdot (\eta - \xi) \geqslant 0. \tag{1.3}$$

We assume moreover that $u \mapsto b(x,u)$ is monotonic and that either $u \mapsto b(x,u)$ is strictly monotonic or the equality in (1.3) implies $a(\xi) = a(\eta)$. This latter condition is a sort of *mild* ellipticity assumption; it is fulfilled for instance if a is the gradient of a convex, $\mathcal{C}^1$ function $f : \mathbb{R}^n \to \mathbb{R}$, which is quite common for the problems of the Calculus of Variations arising from the convexification of a non-convex Lagrangian. The main difficulties here are the non-uniqueness of the solutions to the Dirichlet problem and the fact that affine functions do not satisfy, in general, the Comparison Principles from above or from below.

In the first part of the paper we thoroughly study the set of the solutions, no more unique, to the Dirichlet problem associated to $Lv = 0$ and show in particular under some natural growth condition on a and b that, given a boundary datum ϕ, there is a maximal and a minimal solution to the Dirichlet problem $Lv = 0$, $v = \phi$ on $\partial\Omega$. We formulate some Comparison Principles that, as far as we know, are new in the case where a is not strictly monotonic; we refer to [8, Theorems 2.4.1, 3.4.1] for a statement of the Comparison Principle fore more general operators in divergence form under the more restrictive strict ellipticity assumption on a. We exhibit, in the homogeneous case, a new class of solutions to $Lv = 0$, depending on the level sets of a, that satisfy the Comparison Principle.

We then establish the fact that if ω is any modulus of continuity and u is the maximum or the minimum of the solutions to $Lv = 0$ such that

$$\forall \gamma \in \partial\Omega \quad \big|u(x) - \phi(\gamma)\big| \leqslant \omega\big(|x - \gamma|\big) \quad \text{a.e. } x$$

then $|u(y) - u(x)| \leqslant \omega(|y - x|)$ for a.e. x, y. More precisely the monotonicity of b is enough if b does not depend on x, otherwise we require a monotonicity-like assumption that involves the two variables x, u. This result is well known when $b(x,u) = 0$ and u belongs to the class of a-harmonic functions [9, Lemma 6.47] where the field a needs to satisfy a homogeneity assumption that we do not make here. The result is also the Pde's counterpart of the so-called Haar–Radò theorem for the Lipschitz minimizers of (P) that we recently extended in [10].

The theorems established here, though similar to the corresponding ones in the Calculus of Variations, require some arguments that are new and strictly related to the structure of the partial differential operator.

In the last part of the paper we extend the (BSC) by replacing affine functions with functions depending on the level sets of the field a: it turns out that in the case where a is not strictly monotonic, the class of the boundary data that satisfy this new *Generalized* (BSC) is wider than the class of functions that satisfy the (BSC). To clarify this statement we just mention some facts that make the difference. The (BSC) is a quite restrictive condition: among other properties it forces the domain to be convex. On the other hand once a level set of a contains a ball centered in the origin it turns out that every Lipschitz function of a suitable rank satisfies the Generalized (BSC) with no convexity requirement on the domain Ω.

We then apply the Haar–Radò type theorem in the homogeneous case, i.e. $Lv = -\operatorname{div} a(\nabla v)$ to obtain some regularity results of the solutions to $Lv = 0$. By assuming the natural growth assumptions on the field a from below and from above and the mild ellipticity condition (1.3) where the equality holds if and only if $a(\xi) = a(\eta)$ we prove that every solution to $Lv = 0$ in $\phi + W_0^{1,p}(\Omega)$ is locally Lipschitz whenever ϕ satisfies a unilateral Generalized (BSC). It must be said here that with respect to the analogous result of [7] we drop the uniform ellipticity assumption at the price of a growth condition from above, that is not assumed there. Finally, when ϕ is Lipschitz, we show that every solution to $Lv = 0$ in $\phi + W_0^{1,p}(\Omega)$ is globally Hölder continuous and we explicitly compute its Hölder order.

The results of this part are well established, even for more general classes of operators, under some different structure assumptions: we mention again the a-harmonic functions where the field a needs to be homogeneous and strictly monotonic or the classical results of Lieberman [11] where the solution is assumed to be continuous up to the boundary, smooth in the interior together with some strong ellipticity assumptions. The methods of [11] are also inspired by the minimal surface problem but the construction of barriers there does strongly rely on the uniform ellipticity of the operator.

2. Notation and setting

If v and w are functions then $v \wedge w$ (resp. $v \vee w$) stands for the pointwise minimum (resp. maximum) of v and w. The scalar product in $\mathbb{R}^n$ is denoted by "$\cdot$".

Definition 2.1 *(Modulus of continuity).* A modulus of continuity is a positive continuous function $\omega : [0, +\infty[$ such that $\omega(0) = 0$. A real-valued function ϕ on a set X is ω-continuous if $|\phi(y) - \phi(x)| \leqslant \omega(|y - x|)$ for all $x, y \in X$.

Definition 2.2 *(Inequalities in the trace sense).* Let $u, v \in W^{1,1}(D)$. We say that $u \leqslant v$ in ∂D *in the trace sense* if $u \wedge v \in u + W_0^{1,1}(D)$ or, equivalently, if $u \vee v$ is in $v + W_0^{1,1}(D)$.

Some basic facts about inequalities in the trace sense can be found in [10].

We consider here the following operator in divergence form

$$Lv = -\operatorname{div} a(\nabla v) + b(x, v).$$

Throughout the paper we will make use of the following assumptions.

Basic Assumption (A). The field $a : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and monotonic, i.e.

$$\forall \xi, \eta \in \mathbb{R}^n \quad \big(a(\eta) - a(\xi)\big) \cdot (\eta - \xi) \geqslant 0; \tag{2.1}$$

moreover

$$\forall \xi, \eta \in \mathbb{R}^n \quad \big(a(\eta) - a(\xi)\big) \cdot (\eta - \xi) = 0 \quad \Leftrightarrow \quad a(\xi) = a(\eta). \tag{2.2}$$

The function $b : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that for a.e. $x \in \Omega$ and

$$\forall u, v \in \mathbb{R} \quad \big(b(x, v) - b(x, u)\big)(v - u) \geqslant 0. \tag{2.3}$$

Remark 2.1. Our assumptions (2.1) and (2.2) weaken the more common strict monotonicity assumption

$$\forall \xi, \eta \in \mathbb{R}^n \quad \big(a(\eta) - a(\xi)\big) \cdot (\eta - \xi) > 0 \quad \text{if } \xi \neq \eta.$$

Under just Assumption (A) the solutions to $Lv = 0$ with a prescribed boundary datum may not be unique. Such solutions turn out to be unique if either (2.1) or (2.3) is strict, i.e. either a is strictly monotonic or $u \mapsto b(x, u)$ is strictly monotonic for each x.

Remark 2.2. Assumption (A) is fulfilled if, for instance, a is the gradient of a convex, $\mathcal{C}^1$ function $f : \mathbb{R}^n \to \mathbb{R}$. Indeed if $(\nabla f(\eta) - \nabla f(\xi)) \cdot (\eta - \xi) = 0$ then f is affine on the segment $[\xi, \eta]$ so that the graph of f contains the segment joining $(\xi, f(\xi))$ to $(\eta, f(\eta))$. Let $z = k \cdot x + d$ be a supporting hyperplane to the epigraph of f containing the segment $[(\xi, f(\xi)), (\eta, f(\eta))]$. Since every point of the segment $[\xi, \eta]$ is a global minimum for $f(x) - (k \cdot x + d)$ it follows that $\nabla f(x) = k$ for every $x \in [\xi, \eta]$.

The next example shows that a field satisfying the above conditions (2.1) and (2.2) is not necessarily the gradient of a convex function.

Example 1. Let $n = 3$ and

$$a(\xi) = T(\xi) + \nabla F(\xi)$$

where

$$T(\xi) = (-\xi_3, 0, \xi_1), \qquad F(\xi) = \xi_1^2 + \xi_3^2, \qquad \xi = (\xi_1, \xi_2, \xi_3).$$

For every $\xi, \eta \in \mathbb{R}^3$ we have

$$\begin{aligned}\big(a(\eta) - a(\xi)\big) \cdot (\eta - \xi) &= (\eta - \xi) \cdot T(\eta - \xi) + \big(\nabla F(\eta) - \nabla F(\xi)\big) \cdot (\eta - \xi)\\ &= \big(\nabla F(\eta) - \nabla F(\xi)\big) \cdot (\eta - \xi) \geqslant 0,\end{aligned}$$

since $h \cdot Th = 0$ for all h and F is convex. Moreover the above inequality turns out to be an equality if and only if

$$\big(\nabla F(\eta) - \nabla F(\xi)\big) \cdot (\eta - \xi) = 2(\eta_1 - \xi_1)^2 + 2(\eta_3 - \xi_3)^2 = 0,$$

i.e. when $\eta_1 = \xi_1$ and $\eta_3 = \xi_3$, in which case $a(\xi) = a(\eta)$. However a is not the gradient of a function, otherwise T would have a potential, contradicting the fact that the field T is not irrotational.

3. Comparison Principles

In this section we present the basic tools to prove our regularity results and a Haar–Radò type theorem. Most of them are a reformulation in the Pde's framework of some analogous result that we proved for minimizers of integral functionals in [10]. However their proofs are not a straightforward modification of the variational ones and need some peculiar techniques that we develop here.

3.1. Basic tools

We recall the notion of sub/supersolution to $Lv = 0$. Here we fix $p \geqslant 1$ and q is the conjugate exponent of p.

Definition 3.1. Let $u \in W^{1,p}(\Omega)$ be such that $a(\nabla u), b(x, u(x)) \in L^q(\Omega)$. We say that u is a *subsolution* to $Lv = 0$ (we write that $Lu \leqslant 0$) in Ω if

$$\forall \varphi \in W_0^{1,p}(\Omega),\ \varphi \geqslant 0 \text{ a.e.} \quad \int_\Omega a(\nabla u)\nabla\varphi + b(x, u)\varphi\, dx \leqslant 0; \tag{3.1}$$

u is a *supersolution* to $Lv = 0$ (we write that $Lu \geqslant 0$) in Ω if

$$\forall \varphi \in W_0^{1,p}(\Omega),\ \varphi \geqslant 0 \text{ a.e.} \quad \int_\Omega a(\nabla u)\nabla\varphi + b(x, u)\varphi\, dx \geqslant 0. \tag{3.2}$$

Finally, u is a *solution* to $Lv = 0$ (i.e. $Lu = 0$) in Ω if

$$\forall \varphi \in W_0^{1,p}(\Omega), \quad \int_\Omega a(\nabla u)\nabla\varphi + b(x, u)\varphi\, dx = 0. \tag{3.3}$$

A subsolution (resp. supersolution) to $Lv = 0$ is said to be *strict* if the inequality (3.1) (resp. (3.2)) is strict whenever φ is non-zero.

The next lemma is a key tool in the proof of the subsequent Comparison Principle.

Lemma 3.1. *Assume that L satisfies Assumption* (A). *Let u, w be such that $Lu \leqslant 0$, $Lw \geqslant 0$ and $u \leqslant w$ on $\partial\Omega$. Then the following statements hold*:

(a) *$a(\nabla u) = a(\nabla w)$ and $b(x, u(x)) = b(x, w(x))$ a.e. on the set*

$$\Sigma = \big\{x \in \Omega\colon\ u(x) > w(x)\big\};$$

(b) *$L(u \wedge w) \leqslant 0$ and $L(u \vee w) \geqslant 0$;*
(c) *$L(u \wedge w) = 0$ if $Lu = 0$ and $L(u \vee w) = 0$ if $Lw = 0$.*

Proof. (a) Let $\Sigma = \{x \in \Omega\colon\ u(x) > w(x)\}$. Since, by taking $(u - w)^+$ as a test function

$$\int_\Omega a(\nabla u) \cdot \nabla(u - w)^+ + b(x, u)(u - w)^+\, dx \leqslant 0$$

and

$$\int_\Omega a(\nabla w) \cdot \nabla(u - w)^+ + b(x, w)(u - w)^+\, dx \geqslant 0$$

then

$$\int_\Sigma \big(a(\nabla u)-a(\nabla w)\big)\cdot(\nabla u-\nabla w)+\big(b(x,u)-b(x,w)\big)(u-w)\,dx\leqslant 0$$

so that (A) implies that

$$\big(a(\nabla u)-a(\nabla w)\big)\cdot(\nabla u-\nabla w)=0,\qquad \big(b(x,u)-b(x,w)\big)(u-w)=0\quad\text{a.e. on }\Sigma:$$

again (A) yields $a(\nabla u)=a(\nabla w)$ and $b(x,u)=b(x,w)$ a.e. on Σ.

We show now that $u\vee w$ is a supersolution to $Lv=0$. If $\varphi\in W_0^{1,p}(\Omega)$, $\varphi\geqslant 0$ a.e. then by (a) we have

$$\begin{aligned}\int_\Omega a\big(\nabla(u\vee w)\big)\cdot\nabla\varphi+b(x,u\vee w)\varphi\,dx&=\int_{u\leqslant w} a(\nabla w)\cdot\nabla\varphi+b(x,w)\varphi\,dx+\int_\Sigma a(\nabla u)\cdot\nabla\varphi+b(x,u)\varphi\,dx\\&=\int_{u\leqslant w} a(\nabla w)\cdot\nabla\varphi+b(x,w)\varphi\,dx+\int_\Sigma a(\nabla w)\cdot\nabla\varphi+b(x,w)\varphi\,dx\\&=\int_\Omega a(\nabla w)\cdot\nabla\varphi+b(x,w)\varphi\,dx\geqslant 0.\end{aligned}$$

Notice that the last inequality is actually an equality if $Lw=0$, proving the parts of claims (b) and (c) concerning $u\vee w$. The statements concerning $u\wedge w$ follow similarly. □

In the next result we give some conditions ensuring that $u\leqslant w$ a.e. on Ω once $Lu\leqslant 0$, $Lw\geqslant 0$ and $u\leqslant w$ on $\partial\Omega$. The first of them is the strict monotonicity of a or b: in this case the conclusion is well known and we write it here just for the convenience of the reader. We underline that the main argument here is the uniqueness of the solutions to $Lv=0$ for a prescribed boundary datum. In the general case the Comparison Principle does not hold for arbitrary solutions to $Lv=0$: an example of this situation can be found in [12] in a variational setting. We show in Theorem 3.1(iii) that the minimum and the maximum of the solutions to $Lv=0$ do still satisfy the Comparison Principle.

Definition 3.2. We say that $u\in W^{1,p}(\Omega)$ is the *maximum* (resp. *minimum*) of the solutions to $Lv=0$ if $Lu=0$ and $v\leqslant u$ (resp. $v\geqslant u$) a.e. for every $v\in u+W_0^{1,p}(\Omega)$ satisfying $Lv=0$.

We underline that these solutions do both trivially exist in the case where the solutions to $Lv=0$ are unique. We will show in Section 4 that they still exist if L satisfies some suitable growth conditions.

Theorem 3.1 *(Comparison Principle for extremal solutions). Assume that L satisfies Assumption* (A)*. Let u, w be such that $Lu\leqslant 0$, $Lw\geqslant 0$ and $u\leqslant w$ on $\partial\Omega$. Then $u\leqslant w$ a.e. on Ω if just one of the following assumptions holds*:

(i) *the field a is strictly monotonic or the function $u\mapsto b(x,u)$ is strictly monotonic for a.e. x*;
(ii) *u is a strict subsolution or w is a strict supersolution to $Lv=0$*;
(iii) *u is the minimum of the solutions or w is the maximum of the solutions to $Lv=0$.*

Proof. (i) Claim (a) of Lemma 3.1 implies that $a(\nabla u)=a(\nabla w)$ and $b(x,u(x))=b(x,w(x))$ a.e. on $\Sigma=\{x\in\Omega:\ u(x)>w(x)\}$. The strict monotonicity of a (resp. of $b(x,\cdot)$) yields that $\nabla u=\nabla w$ (resp. $u=w$) a.e. on Σ: in both cases we obtain that $(u-w)^+=0$ a.e. on Ω.

(ii) Assume that $Lu<0$ and that by contradiction $u>w$ on a non-negligible set Σ. Then by taking $(u-w)^+$ as a test function we get

$$\int_\Sigma a(\nabla u)\cdot\nabla(u-w)^+ + b(x,u)(u-w)^+\,dx<0.$$

Now Lemma 3.1 implies that $a(\nabla u)=a(\nabla w)$ and $b(x,u(x))=b(x,w(x))$ a.e. on $\Sigma=\{x\in\Omega:\ u(x)>w(x)\}$ and thus

$$\int_\Omega a(\nabla w)\cdot\nabla(u-w)^+ + b(x,w)(u-w)^+\,dx<0,$$

contradicting the fact that $Lw\geqslant 0$; it follows that $u\leqslant w$ a.e. The case where w is a strict supersolution follows similarly.

(iii) Assume that u is the minimum of the solutions to $Lv = 0$. Since by (c) of Lemma 3.1 $u \wedge w$ is still a solution to $Lu = 0$ then the minimality of u yields $u \leqslant u \wedge w$ so that $u \leqslant w$ a.e. The case where w is the maximum of the solutions to $Lu = 0$ follows similarly. □

Remark 3.1. As a particular case of Theorem 3.1(iii) the Comparison Principle holds whenever either u or w are unique.

3.2. Behavior of solutions with respect to translations

Let now

$$\omega : [0, +\infty[\longrightarrow [0, +\infty[$$

be any positive modulus of continuity. We consider the following monotonicity assumption on the function b.

Assumption (B_ω).

$$\forall x, y \in \mathbb{R}^n,\ \forall u, v \in \mathbb{R} \quad v \geqslant u + \omega\big(|y - x|\big) \quad \Rightarrow \quad b(y, v) \geqslant b(x, u).$$

Remark 3.2. Notice that (B_ω) implies (2.3): indeed if $x \in \Omega$ then $\omega(0) = 0 = x - x$ so that if $v \geqslant u$ then $v \geqslant u + \omega(0)$ and the validity of (B_ω) yields $b(x, v) \geqslant b(x, u)$. Moreover if $b(x, u) = b(u)$, namely b does not depend on x, then (B_ω) is fulfilled if and only if b is increasing.

The following theorem states that, in the case we are considering, the property of being a subsolution or supersolutions is preserved under suitable translations. For any $h \in \mathbb{R}^n$ we set $\Omega_h = h + \Omega$.

Theorem 3.2. *Let ω be a modulus of continuity, $h \in \mathbb{R}^n$ and assume that L satisfies Assumptions* (A) *and* (B_ω). *Let u be a subsolution of $Lu = 0$. Then $u(y - h) - \omega(|h|)$ is a subsolution of $Lv = 0$ on Ω_h. Analogously, if u is a supersolution of $Lv = 0$ on Ω then $u(y + h) + \omega(|h|)$ is a supersolution of $Lv = 0$ on $\Omega_h := h + \Omega$.*

Proof. Let u be a subsolution of $Lv = 0$ and set $c = \omega(|h|)$, $w(y) = u(y - h) - c$. Let $\varphi \in W_0^{1,p}(\Omega_h)$ be positive a.e.; the change of variables $y = x + h$ yields

$$\begin{aligned} I &:= \int_{\Omega_h} a\big(\nabla w(y)\big) \cdot \nabla\varphi(y) + b\big(y, w(y)\big)\varphi(y)\, dy \\ &= \int_{\Omega} a\big(\nabla u(x)\big) \cdot \nabla\psi(x) + b\big(x + h, u(x) - c\big)\psi(x)\, dx \end{aligned}$$

where we set $\psi(x) = \varphi(x + h)$, a function of $W_0^{1,p}(\Omega)$. Therefore $I = \Xi + \Pi$ with

$$\begin{aligned} \Xi &= \int_{\Omega} a\big(\nabla u(x)\big) \cdot \nabla\psi(x) + b\big(x, u(x)\big)\psi(x)\, dx, \\ \Pi &= \int_{\Omega} \Big(b\big(x + h, u(x) - c\big) - b\big(x, u(x)\big)\Big)\psi(x)\, dx. \end{aligned}$$

Now $\Xi \leqslant 0$ since u is a subsolution of $Lv = 0$; moreover since $u(x) - (u(x) - c) = c \geqslant c$ then (B_ω) implies that $b(x, u(x)) \geqslant b(x + h, u(x) - c)$ so that $\Pi \leqslant 0$: it follows that $I \leqslant 0$, i.e. $w(y) = u(y - h) - \omega(|h|)$ is a subsolution. The part of the claim concerning supersolutions follows similarly. □

3.3. A Haar–Radò type theorem

The next result is in the flavor of the well-known properties that hold both in the Calculus of Variations and in the Pde's setting. In the first case it is known as Haar–Radò theorem and it holds for Lipschitz minimizers of strictly convex functionals of the gradient, whereas for differential equation can be found in [9, Lemma 6.47] for a-harmonic functions. The proof there is based on the particular structure of the operator (strict ellipticity and homogeneity in the gradient variable, neither of them is assumed here) that allows the use of Harnack inequality in the interior of the domain. Our approach is based on the validity of the Comparison Principles stated before. Our proof is directly inspired by our recent generalization of Haar–Radò theorem in the Calculus of Variations [10]; we give it here for the sake of completeness.

Theorem 3.3 *(Haar–Radò type). Let $\omega : [0, +\infty[\to [0, +\infty[$ be a modulus of continuity, ϕ be a function in $W^{1,p}(\Omega)$ which is ω-continuous on $\overline{\Omega}$. Let L satisfy Assumptions* (A), (B_ω) *and u be the maximum or the minimum of the solutions to $Lv = 0$ on $\phi + W_0^{1,p}(\Omega)$. Assume that one of the following assumptions holds:*

(H_1) *$u, \phi \in \mathcal{C}(\overline{\Omega})$ and*

$$\forall \gamma \in \partial\Omega,\ \forall x \in \Omega \quad \bigl|u(x) - \phi(\gamma)\bigr| \leqslant \omega\bigl(|x - \gamma|\bigr); \tag{3.4}$$

(H_2) *$\Omega \cap \Omega_h$ is regular for all $h \in \mathbb{R}^n$; moreover*

$$\forall \gamma \in \partial\Omega \quad \bigl|u(x) - \phi(\gamma)\bigr| \leqslant \omega\bigl(|x - \gamma|\bigr) \quad \text{a.e. } x; \tag{3.5}$$

(H_3) *$\Omega \cap \Omega_h$ is regular for all $h \in \mathbb{R}^n$ and there exist $\ell_1, \ell_2 \in \phi + W_0^{1,1}(\Omega)$ that are ω-continuous on $\overline{\Omega}$ and such that*

$$\ell_1(x) \leqslant u(x) \leqslant \ell_2(x) \quad \text{a.e. on } \Omega. \tag{3.6}$$

Then $|u(y) - u(x)| \leqslant \omega(|y - x|)$ for every Lebesgue points x and y of u.

Proof. Assume that u is the maximum of the solutions to $Lv = 0$. We know that, as in [10, Lemma 4.1], $u_h - \omega(|h|) \leqslant u$ on $\partial(\Omega \cap \Omega_h)$ in the trace sense in all the three cases (H_1), (H_2), (H_3). By Theorem 3.2 $u_h - \omega(|h|)$ is a subsolution of $Lv = 0$ on Ω_h and thus on $\Omega \cap \Omega_h$ whereas u is still the maximum of the solutions to $Lv = 0$ on $\Omega \cap \Omega_h$. The Comparison Principle (Theorem 3.1) implies that $u_h - \omega(|h|) \leqslant u$ a.e. on $\Omega \cap \Omega_h$. Now let x, y be two Lebesgue points of u and let $r > 0$ be such that $B_r(x)$ and $B_r(y)$ are contained in Ω. Let $h = y - x$; since $u(z + h) \leqslant u(z) + \omega(|h|)$ for a.e. $z \in B_r(x)$, it turns out by integration on balls of radius r and then passing to the limit as r tends to 0 that $u(y) - u(x) \leqslant \omega(|y - x|)$; proving the claim. The case where u is the minimum of the solutions follows similarly. □

Remark 3.3. In (H_2) it is enough to assume that there exists $\delta > 0$ such that $\Omega \cap (h + \Omega)$ is regular for all $|h| \leqslant \delta$ and that (3.5) holds for every γ and a.e. x with $|x - \gamma| \leqslant \delta$. Indeed, proceeding as in the proof of Theorem 3.3, one obtains that $|u(y) - u(x)| \leqslant \omega(|y - x|)$ a.e. for every Lebesgue points x, y with $|y - x| \leqslant \delta$. Let now x, y be any Lebesgue points; set m to be the integer part of $|x - y|/\delta$ and

$$z_i = x + i\delta \frac{y - x}{|y - x|}, \quad i = 0, \ldots, m.$$

Then

$$u(y) - u(x) = u(y) - u(z_m) + \sum_{i=0}^{m-1} \bigl(u(z_{i+1}) - u(z_i)\bigr) \leqslant m\omega(\delta) + \omega\bigl(|y - z_m|\bigr).$$

Now ω being increasing we get

$$u(y) - u(x) \leqslant (m + 1)\omega\bigl(|y - x|\bigr)$$

and the conclusion follows since $m \leqslant \operatorname{diam} \Omega/\delta$.

4. The set of solutions to $Lv = 0$

In this section we assume $p > 1$ and we posit that L satisfies the Basic Assumption (A) and moreover, the following *growth condition*.

Growth Assumption (G).

$$\forall \xi \in \mathbb{R}^n \quad a(\xi) \cdot \xi \geqslant \alpha|\xi|^p, \qquad \bigl|a(\xi)\bigr| \leqslant \beta|\xi|^{p-1} + r, \tag{4.1}$$

$$\forall u \in \mathbb{R} \quad \bigl|b(x, u)\bigr| \leqslant \gamma|u|^{p-1} + s(x) \quad \text{a.e. in } \Omega \tag{4.2}$$

for some $\alpha, \beta, \gamma > 0$, $r \geqslant 0$ and $s \in L^q(\Omega)$ where q is the conjugate exponent of p.

Remark 4.1. The existence of a solution to $Lv = 0$ follows from the Browder–Minty theorem [13] if the function b is equal to zero or the constants, α, β, γ are chosen in such a way that the operator L is coercive.

Definition 4.1. A *level set* for the field $a:\mathbb{R}^n\to\mathbb{R}^n$ is

$$F_\xi=\{\eta\in\mathbb{R}^n:\ a(\eta)=a(\xi)\}$$

for some $\xi\in\mathbb{R}^n$.

The next intermediate result gives some information on the convexity and boundedness of the level sets of a.

Lemma 4.1. *Let F_ξ be a* level set *for the field a. The set F_ξ is closed and moreover*

(a) *if a satisfies* (2.1) *and* (2.2) *of Assumption* (A) *then F_ξ is convex*;
(b) *if a satisfies* (4.1) *of Assumption* (G) *there exist $c>0$ and $r\in\mathbb{R}$ such that*

$$\forall\eta\in F_\xi\quad |\eta|\leqslant c|\xi|+r. \tag{4.3}$$

Proof. (a) Let $\eta_1,\eta_2\in F_\xi$ and $\zeta=\lambda\eta_1+(1-\lambda)\eta_2$ for some $\lambda\in]0,1[$. Since, by (2.1), we have

$$0\leqslant\big(a(\zeta)-a(\eta_1)\big)\cdot(\zeta-\eta_1),\qquad 0\leqslant\big(a(\zeta)-a(\eta_2)\big)\cdot(\zeta-\eta_2),$$

it turns then out by replacing ζ with $\lambda\eta_1+(1-\lambda)\eta_2$ that

$$0\leqslant(1-\lambda)\big(a(\zeta)-a(\eta_1)\big)\cdot(\eta_2-\eta_1),\qquad 0\leqslant\lambda\big(a(\zeta)-a(\eta_1)\big)\cdot(\eta_1-\eta_2).$$

Since $a(\eta_1)=a(\eta_2)$ it follows that $(a(\zeta)-a(\eta_1))\cdot(\zeta-\eta_1)=0$: (2.2) yields the conclusion.

(b) The claim follows directly from the inequalities

$$\alpha|\eta|^p\leqslant a(\eta)\cdot\eta=a(\xi)\cdot\eta\leqslant\big(\beta|\xi|^{p-1}+r\big)|\eta|.\qquad\Box$$

Theorem 4.1. *Assume that L satisfies Assumption* (A). *Let ϕ in $W^{1,p}(\Omega)$ be such that the set X of solutions to $Lv=0$ in $\phi+W_0^{1,p}(\Omega)$ is non-empty. The following conclusions hold*:

(a) *If $u\in X$ and $v\in\phi+W_0^{1,p}(\Omega)$, then $v\in X$ if and only if $a(\nabla v)=a(\nabla u)$ and $b(x,u)=b(x,v)$ a.e. in Ω*;
(b) *if $u,v\in X$ then $u\vee v, u\wedge v\in X$*;
(c) *X is convex*;
(d) *If* (4.3) *holds then the set X is weakly compact.*

Proof. (a) It is obvious that if $a(\nabla v)=a(\nabla u)$ and $b(x,u)=b(x,v)$ a.e. in Ω then $Lv=Lu=0$. Conversely assume that $v\in X$; by taking $\varphi=(v-u)^+$ as a test function we obtain that

$$\int\limits_{\{v\geqslant u\}}\big(a(\nabla v)-a(\nabla u)\big)\cdot(\nabla v-\nabla u)+\big(b(x,v)-b(x,u)\big)(v-u)\,dx=0$$

whereas, by taking $\varphi=(u-v)^+$ as a test function we obtain that

$$\int\limits_{\{v\leqslant u\}}\big(a(\nabla v)-a(\nabla u)\big)\cdot(\nabla v-\nabla u)+\big(b(x,v)-b(x,u)\big)(v-u)\,dx=0$$

and thus

$$\int\limits_{\Omega}\big(a(\nabla v)-a(\nabla u)\big)\cdot(\nabla v-\nabla u)+\big(b(x,v)-b(x,u)\big)(v-u)\,dx=0.$$

Since, from Assumption (A), the above integrands are positive then

$$\big(a(\nabla v)-a(\nabla u)\big)\cdot(\nabla v-\nabla u)=0,\qquad\big(b(x,v)-b(x,u)\big)(v-u)=0\quad\text{a.e.},$$

thus $a(\nabla u)=a(\nabla v)$ and $b(x,v)=b(x,u)$ a.e.

(b) If $u,v\in X$ then $a(\nabla u)=a(\nabla v)$ and $b(x,u)=b(x,v)$ a.e. so that $a(\nabla(u\wedge v))=a(\nabla(u\vee v))=a(\nabla u)$ and analogously $b(x,u\wedge v)=b(x,u\vee v)=b(x,u)$ a.e.: (a) implies that both $u\wedge v, u\vee v\in X$.

(c) Let $u,v\in X$ and set $w=\lambda u+(1-\lambda)v$ for some $\lambda\in[0,1]$. By (a) we know that $a(\nabla u)=a(\nabla v)$ and $b(x,u)=b(x,v)$ a.e.; it thus follows from Lemma 4.1(a) that $a(\nabla w)=a(\nabla u)$ a.e. Moreover for a.e. $w(x)$ belongs to the segment joining $u(x)$ to $v(x)$: the monotonicity of $b(x,\cdot)$ then implies that $b(x,u(x))=b(x,w(x))=b(x,v(x))$: (a) yields the claim.

(d) Let $c > 0$ and r be such that $|\eta| \leqslant c|\xi| + r$ for all $\eta \in F_\xi$. Let $(u_k)_k$ be a sequence in X. Then by (a) we have $a(\nabla u_k) = a(\nabla u_1)$ a.e. so that $|\nabla(u_k(x) - u_1(x))| \leqslant c|\nabla u_1| + r$ a.e. for every k. It follows that the sequence $(u_k - u_1)_k$ is weak precompact in $W^{1,p}(\Omega)$ so that a subsequence does weakly converge to a function v in $W^{1,p}(\Omega)$: set $u = u_1 + v$. Since $\nabla u_k \rightharpoonup \nabla u$ in $L^p(\Omega)$, by Mazur's lemma there is a sequence $(\nabla w_k)_k$ of convex combination of ∇u_k that converges strongly to ∇u in $L^p(\Omega)$; modulo a subsequence we may assume that both the convergence of ∇u_k to ∇u and of u_k to u hold a.e. Notice that since the level sets of a are convex and $\nabla u_k(x) \in F_{\nabla u_1(x)}$ a.e. then $\nabla w_k(x) \in F_{\nabla u_1(x)}$ a.e. for every k. At every point x of convergence we thus have that $\nabla u(x) \in F_{\nabla u_1(x)}$ or equivalently $a(\nabla u(x)) = a(\nabla u_1(x))$. Moreover the continuity of $b(x,\cdot)$ implies that $b(x, u_k(x)) \to b(x, u(x))$ a.e. It follows then by (a) that $u \in X$ proving that the sequence $(u_k)_k$ has a subsequence that does weakly converge in X. □

We are now in the position to prove the existence of the minimum and the maximum of the solutions to $Lv = 0$ with a prescribed boundary datum.

Theorem 4.2 *(Existence of extremal solutions). Assume that L satisfies Assumption* (A) *and that* (4.3) *holds. Let ϕ be such that the set X of solutions to $Lv = 0$ in $\phi + W_0^{1,p}(\Omega)$ is non-empty. Then there exist $u^-, u^+ \in X$ satisfying $u^- \leqslant u \leqslant u^+$ a.e. for all $u \in X$ with $Lu = 0$. In particular u^+ and u^- exist if L satisfies the Growth Assumption* (G)*.*

Proof. From the closure and the convexity of X together with the separability of $W^{1,p}(\Omega)$ there is a dense sequence $(u_k)_k$ in X. For every $k \in \mathbb{N}$ set $v_k = u_1 \vee \cdots \vee u_k$ and let u^+ be the pointwise limit of v_k. From the weak compactness of X there is $w \in X$ such that v_k converges weakly to w; thus v_k converges strongly to w in $L^p(\Omega)$ so that $u^+ = w \in X$. Clearly $u^+ \geqslant u$ for every $u \in X$. The existence of u^- follows similarly. □

As an application we show that the solutions to $-\operatorname{div} a(\nabla u) = 0$ are bounded. The local boundedness of solutions to nonhomogeneous quasilinear elliptic equations is in general a difficult matter, see [8] for some recent results on the subject; in the case of our simplest operators the existence of a global bound follows immediately from our Comparison Principle, even without assuming any growth condition.

Theorem 4.3 *(Boundedness of solutions). Assume that $Lu = -\operatorname{div} a(\nabla u)$ satisfies* (2.1) *and* (2.2) *of Assumption* (A)*. Assume moreover that either a fulfills condition* (4.3)*. Then if $\phi \in L^\infty(\partial\Omega)$ any solution to $Lu = 0$ subject to $u = \phi$ on $\partial\Omega$ is bounded.*

Proof. Let $M = \|\phi\|_\infty$. If the set of solutions to $Lu = 0, u = \phi$ is empty there is nothing to prove. Otherwise let u be a solution to our problem. Let u^+ be the maximal solution to $Lu = 0$ with $u = M$ on $\partial\Omega$ and u^- be the minimal solution to $Lu = 0$ with $u = -M$ on $\partial\Omega$: these functions exist in view of Theorem 4.2 since constants are solutions to $Lu = 0$. Notice that both ∇u^- and ∇u^+ belong to F_0 which is bounded due to (4.3), thus u^- and u^+ are bounded. Now $u^- \leqslant u \leqslant u^+$ on $\partial\Omega$: the Comparison Principle Theorem 3.1 yields the conclusion. □

5. The generalized (BSC)

From now on we consider the homogeneous case

$$Lv = -\operatorname{div} a(\nabla v)$$

and we assume that the field a satisfies the Basic Assumption (A).

5.1. A class of functions that satisfies the Comparison Principle

We consider the translates of the support functions of a compact and convex set, first introduced by Cellina [12] in the framework of the Calculus of Variations to deal with non-strictly convex problems.

Definition 5.1 *(A class of functions).* Whenever F is a compact and convex subset of $\mathbb{R}^n$ and $x_0 \in \mathbb{R}^n$ we consider the functions

$$h^+_{F,x_0}(x) = \max\{\xi \cdot (x - x_0)\colon \xi \in F\}, \qquad h^-_{F,x_0}(x) = \min\{\xi \cdot (x - x_0)\colon \xi \in F\}.$$

Example 2. Let F be the unit ball. Then, for all x_0,

$$h^+_{F,x_0}(x) = |x - x_0|, \qquad h^-_{F,x_0}(x) = -|x - x_0|.$$

It is worth mentioning that the functions just defined are Lipschitz, $\nabla h^\pm_{F,x_0} \in F$ a.e. and that

$$h^\pm_{F,x_0}(x) = \nabla h^\pm_{F,x_0}(x) \cdot (x - x_0) \quad \text{a.e.} \tag{5.1}$$

as it can be easily seen from the properties of the support function to a set [14] or see [12] for a direct proof; they are nothing more than affine when F is reduced to a single point. We show now that these functions satisfy the Comparison Principle with respect to any other minimizer (not just the minimum or the maximum ones). The proposition is the reformulation in this Pde's setting of a result by Cellina in [12] and its refinements in [15,16].

Proposition 5.1. *Assume that L satisfies Assumption* (A) *and let F be a compact level set for a. For every $x_0 \in \mathbb{R}^n$ and $c \in \mathbb{R}$ the functions $c + h^{\pm}_{F,x_0}$ are solutions to $-\operatorname{div} a(\nabla v) = 0$ in Ω. Moreover if $x_0 \notin \Omega$ they satisfy the Comparison Principle*:

$$u \in \left(c + h^{\pm}_{F,x_0}\right) + W^{1,1}_0(\Omega), \qquad Lu \leqslant 0, \qquad u \leqslant c + h^{\pm}_{F,x_0} \quad \text{on } \partial\Omega \quad \Rightarrow \quad u \leqslant c + h^{\pm}_{F,x_0} \quad \text{a.e. on } \Omega,$$

$$u \in \left(c + h^{\pm}_{F,x_0}\right) + W^{1,1}_0(\Omega), \qquad Lu \geqslant 0, \qquad u \geqslant c + h^{\pm}_{F,x_0} \quad \text{on } \partial\Omega \quad \Rightarrow \quad u \geqslant c + h^{\pm}_{F,x_0} \quad \text{a.e. on } \Omega.$$

Proof. Let $F = \{\eta \in \mathbb{R}^n \colon a(\eta) = a(\xi)\}$ for some ξ. The fact that $c + h^{\pm}_{F,x_0}$ are solutions follows immediately since

$$a\left(\nabla\left(c + h^{\pm}_{F,x_0}\right)\right) = a(\xi)$$

is a constant. Assume now without restriction that $c = 0$. Set $h = h^{+}_{F,x_0}$ and let $u \in h + W^{1,1}_0(\Omega)$ be a solution to $Lv = 0$. By Theorem 4.1 we obtain that $a(\nabla u) = a(\nabla h)$ a.e. so that $\nabla u \in F$ a.e.; it follows from (5.1) that

$$\nabla u(x) \cdot (x - x_0) \leqslant \nabla h^{+}_{F,x_0}(x) \cdot (x - x_0) = h(x) \quad \text{a.e.}$$

and therefore, if we set $\psi = h - u$, we have

$$\psi \in W^{1,1}_0(\Omega), \quad \nabla\psi(x) \cdot (x - x_0) \geqslant 0 \quad \text{a.e. on } \Omega.$$

We resume here the same reasoning that was carried on in [17]: there is a representative ψ^* of ψ that is zero on $\partial\Omega$ and such that ψ^* is absolutely continuous on a.e. line through x_0 and such that, for a.e. $x \in \Omega$,

$$\forall t \quad \frac{d}{dt}\psi^*\left(x_0 + t(x - x_0)\right) = \nabla\psi\left(x_0 + t(x - x_0)\right) \cdot (x - x_0) \geqslant 0$$

so that ψ^* increases along a.e. line from x_0. Since $\psi^* = 0$ on $\partial\Omega$ it follows that ψ^* does actually vanish along these lines, so that $\psi^* = 0$ a.e. on Ω. Thus $\psi = 0$ a.e. on Ω and $u = h$, so that $h = h^{+}_{F,x_0}$ is the only solution with such a boundary datum. The same reasoning applies to h^{-}_{F,x_0}. Remark 3.1 yields the conclusion. $\square$

5.2. Bounded Slope Conditions

We first recall the Bounded Slope Condition introduced by Hartmann and Stampacchia in [1].

Definition 5.2 *(BSC).* The function ϕ satisfies the *Bounded Slope Condition* of rank $M \geqslant 0$ if for every $\gamma \in \partial\Omega$

$$\exists z^{-}_{\gamma} \in \mathbb{R}^n, \ \left|z^{-}_{\gamma}\right| \leqslant M, \ \forall \gamma' \in \partial\Omega \quad \phi(\gamma) + z^{-}_{\gamma} \cdot \left(\gamma' - \gamma\right) \leqslant \phi\left(\gamma'\right), \tag{5.2}$$

$$\exists z^{+}_{\gamma} \in \mathbb{R}^n, \ \left|z^{+}_{\gamma}\right| \leqslant M, \ \forall \gamma' \in \partial\Omega \quad \phi(\gamma) + z^{+}_{\gamma} \cdot \left(\gamma' - \gamma\right) \geqslant \phi\left(\gamma'\right). \tag{5.3}$$

Remark 5.1. We remind that ϕ satisfies the (BSC) if and only if it is the restriction of a convex function and of a concave function, both defined on $\mathbb{R}^n$ and globally Lipschitz. Under a uniform convexity assumption on the domain any $\mathcal{C}^2$ function satisfies the (BSC) [18]. The (BSC) is a quite restrictive condition: it forces for instance the function ϕ to be affine on the flat parts of $\partial\Omega$ and Ω to be convex.

Recently, some new conditions that are less restrictive than the (BSC) appeared in the literature for problems of the Calculus of Variations depending on the gradient. The Lower (resp. Upper) (BSC) was introduced by Clarke: it requires the validity of just (5.2) (resp. (5.3)), which turns out in [4] to be sufficient to obtain the local Lipschitz continuity of the minimizers of strictly convex functionals. A generalized (BSC) was introduced by Cellina, where the functions $c + h^{\pm}_{F,x_0}$ defined above replace affine functions in the (BSC): when the sets F are the projections onto $\mathbb{R}^n$ of the faces of the epigraph of the Lagrangian the condition turns out to be sufficient in [19] to obtain the Lipschitz continuity of the minimizers. The generalized (BSC) is particularly suitable and interesting when the Lagrangian is not strictly convex, since in this case some of the faces of its epigraph are not reduced to a point, so that the functions $h^{\pm}_{F,x_0}$ are not affine.

Let us first formulate the definition of the Generalized (BSC) in this context in this framework. We consider as above the functions $c + h^{\pm}_{F,x_0}$, where the sets F are level sets of the field a, no more related to the faces of a Lagrangian. This explains why the proof of the subsequent results differs from their analogous versions that have been established in [15,19].

We recall that we consider the operator $Lv = -\operatorname{div} a(\nabla v)$, where $a : \mathbb{R}^n \to \mathbb{R}^n$ satisfies Assumption (A).

Definition 5.3 *(Generalized (BSC) or (GBSC)).* The pair (ϕ, a) satisfies the *Generalized* (*BSC*) of rank $M \geqslant 0$ if for every $\gamma \in \partial\Omega$:

(i) there exists a level set F^- for a, contained in a ball of radius M, such that

$$\forall \gamma' \in \partial\Omega \quad \phi(\gamma) + h^-_{F^-,\gamma}\big(\gamma'\big) \leqslant \phi\big(\gamma'\big); \tag{5.4}$$

(ii) there exists a level set F^+ for a, contained in a ball of radius M, such that

$$\forall \gamma' \in \partial\Omega \quad \phi(\gamma) + h^+_{F^+,\gamma}\big(\gamma'\big) \geqslant \phi\big(\gamma'\big). \tag{5.5}$$

The pair (ϕ, a) is said to satisfy the *Generalized Lower* (resp. *Upper*) (*BSC*) if just (5.4) (resp. (5.5)) holds.

Remark 5.2. Opposite to the (BSC) the definition of the Generalized (BSC) involves the field a.

The term “Generalized” in the new (BSC) is motivated by the following result.

Proposition 5.2. *Assume that the field L satisfies Assumptions* (A), (G) *and that* (ϕ, a) *satisfies the Lower* (*resp. Upper*) (*BSC*) *of rank M. Then* ϕ *satisfies the Lower* (*resp. Upper*) *Generalized* (*BSC*) *of a rank depending only on a and M.*

Proof. Assume that for some γ and $z \in \mathbb{R}^n$ with $|z| \leqslant M$ we have

$$\forall \gamma' \in \partial\Omega \quad \phi(\gamma) + z \cdot \big(\gamma' - \gamma\big) \leqslant \phi\big(\gamma'\big).$$

Let F_z be the level set for a defined by

$$F_z = \big\{\eta \in \mathbb{R}^n\colon a(\eta) = a(z)\big\}.$$

Since, clearly, for all $\gamma' \in \partial\Omega$

$$h^-_{F_z,\gamma}\big(\gamma'\big) = \min\big\{\xi \cdot \big(\gamma' - \gamma\big)\colon \xi \in F_z\big\} \leqslant z \cdot \big(\gamma' - \gamma\big)$$

then $\phi(\gamma) + h^-_{F_z,\gamma}(\gamma') \leqslant \phi(\gamma')$. It follows from Lemma 4.1(b) that if $\eta \in F_z$ then

$$|\eta| \leqslant c|z| + r \leqslant cM + r$$

so that F_z is contained in a ball of center 0 and radius depending on a and M, thus proving the validity of the Generalized Lower (BSC) in both the cases. The version of the result for the Upper (BSC) follows similarly. □

Example 3. The Generalized (BSC) is strictly more general than the (BSC). For instance if

$$a(\xi) = \nabla f(\xi) \quad \text{where } f(\xi) = \begin{cases} (|\xi|^2 - 1)^2 & \text{if } |\xi| \geqslant 1; \\ 0 & \text{otherwise,} \end{cases}$$

then the level set F of a containing the origin is the closed unit ball. It follows from Example 2 that $h^+_{F,x_0}(x) = |x - x_0|$ and $h^-_{F,x_0}(x) = -|x - x_0|$. Therefore any Lipschitz function ϕ of rank less or equal than 1 is such that (ϕ, a) satisfies the Generalized (BSC); note that, opposite to what happens when ϕ satisfies the (BSC), the domain may be not convex.

6. Regularity results for the solutions to $-\operatorname{div} \boldsymbol{a}(\nabla v) = 0$

In this section we assume that $Lv = -\operatorname{div} a(\nabla v)$ where the field a satisfies the Basic Assumption (A).

6.1. Lipschitz regularity

It has been proved recently in [20] that if ϕ satisfies the Generalized (BSC) then there is a Lipschitz solution to $Lv = 0$. Solutions are not unique; it is natural to ask whether every solution to $Lv = 0$ in $\phi + W^{1,1}_0(\Omega)$ is Lipschitz.

Theorem 6.1 *(Lipschitz continuity with the Generalized (BSC)). Let* Ω *be an open and bounded subset of* $\mathbb{R}^n$*. Assume that the field L satisfies Assumption* (A) *and that* (ϕ, a) *satisfies the* Generalized (BSC) *of rank M. Assume moreover that there is* $R > 0$ *that bounds the diameter of the level sets that intersect the ball of radius M, i.e.*

$$\forall \eta, \xi \quad a(\eta) = a(\xi), \qquad |\xi| \leqslant M \quad \Rightarrow \quad |\eta - \xi| \leqslant R.$$

Then every solution to $Lv = 0$ *in* $\phi + W^{1,1}_0(\Omega)$ *is Lipschitz of rank less than* $R + M$.

Proof. By [20, Theorem 4.1] there exists a Lipschitz function w of rank less than M that

$$\int_{\Omega} a(\nabla w)\cdot\nabla\eta\,dx=0$$

for every $\eta\in W_0^{1,1}(\Omega)$. If $u\in\phi+W_0^{1,1,}(\Omega)$ is such that $Lu=0$ then, by Theorem 4.1(a) $a(\nabla u)=a(\nabla w)$ a.e. Since $|\nabla w|\leqslant M$ a.e. the condition on the level sets then implies that

$$|\nabla u|\leqslant|\nabla u-\nabla w|+|\nabla w|\leqslant R+M. \qquad \square$$

6.2. Local Lipschitz regularity

We assume now that the field a satisfies the Basic Assumption (A) and, moreover, the Growth Assumption (G). We recall that in this situation the Browder–Minty theorem ensures the existence of a solution to the Dirichlet problem associated to L.

In [4] Clarke introduced the unilateral (BSC) to obtain the local Lipschitz regularity of the minimum of a variational problem of the gradient. We generalized it to the case of non-strictly convex Lagrangian in [19]. The Comparison Principles established here allow us to convert the result in the framework of Pde's. We underline that, beside the interest of this results in itself, it is also a basic tool in the subsequent proof of the Hölder continuity of the solutions to $Lv=0$ (Theorem 6.3).

Theorem 6.2 *(Local Lipschitz continuity). Let Ω be an open, convex and bounded subset of $\mathbb{R}^n$. Assume that L satisfies Assumptions* (A), (4.3) *and that (ϕ,a) fulfills the Lower or the Upper* (GBSC)*. Let u be a solution to $Lv=0$ in $\phi+W_0^{1,p}(\Omega)$. Then u is locally Lipschitz.*

Proof. The proof is similar to those of [4, Theorems 2.1] and [19, Theorem 4.1]. We just point out the new fact that we are dealing with solutions to a Pde instead of minimizers of an integral functional, by showing that the main arguments of these proofs do still work in this setting. Consider first the maximum of the solutions w to $Lv=0$ on $\phi+W_0^{1,p}(\Omega)$.

(i) For $\lambda\in]0,1]$ and $\gamma\in\partial\Omega$ the function $w_\lambda(x)=\lambda w(\frac{x-\gamma}{\lambda}+\gamma)$ is still a solution to $Lv=0$ on $\Omega_\lambda=\lambda(\Omega-\gamma)+\gamma$: it is the greater one among those that share the same boundary datum.

(ii) By Proposition 5.1 the Lower (GBSC) implies that $w(x)\geqslant\phi(\gamma)+h^-_{F^-_\gamma,\gamma}(x)$ for a.e. $x\in\Omega$.

(iii) w is bounded. Indeed, fix $\gamma\in\partial\Omega$: since $w(x)\geqslant\phi(\gamma)+h^-_{F^-_\gamma,\gamma}(x)$ then w is bounded from below. Moreover let $C=\|\phi\|_\infty$ and w_C be the greatest solution to $Lv=0$ with $v=C$ on $\partial\Omega$. By Theorem 4.1 we have $a(0)=a(\nabla C)=a(\nabla w_C)$ so that ∇w_C belongs a.e. to the level set $\{\eta\in\mathbb{R}^n\colon\ a(\eta)=a(0)\}$ which is bounded thanks to (4.3). Thus u_C is Lipschitz and bounded by a constant depending only on C, a and $\operatorname{diam}\Omega$. The proof then proceeds as in [4, Theorem 2.1] showing that w is locally Lipschitz.

(iv) Since u is a solution to $Lv=0$ with the same boundary datum then, by Theorem 4.1 $a(\nabla u)=a(\nabla w)$ a.e.; Lemma 4.1(b) implies that

$$|\nabla u|\leqslant c|\nabla w|+r \quad \text{a.e.}$$

proving that the gradient of u is bounded in any compact subset of Ω. $\square$

Remark 6.1. The existence of a solution to $Lv=0$ with $u=\phi$ on $\partial\Omega$ is ensured for any $p>1$ if one assumes moreover that a satisfies the Growth Assumption (G) or, as it is shown in [7] if a is uniformly elliptic, i.e. there exists $\mu>0$ such that $(a(\eta)-a(\xi))\cdot(\eta-\xi)\geqslant\mu|\eta-\xi|^2$ in the case where $p=2$.

6.3. Local Hölder regularity

Finally in the next theorem we have two statements that are, respectively, the analogues of [6, Theorem 4.5] and [5, Theorem 2].

Theorem 6.3 *(Continuity and Hölder continuity). Let Ω be an open, convex and bounded subset of $\mathbb{R}^n$. Assume that L satisfies Assumptions* (A) *and* (G)*. Let u be the maximum or the minimum of the solution to $Lv=0$ on $\phi+W_0^{1,p}(\Omega)$. Then*

(a) *if ϕ is Lipschitz then u is Hölder continuous in $\overline{\Omega}$ of order $\alpha=\frac{p-1}{n+p-1}$;*
(b) *if ϕ is continuous then u is continuous on $\overline{\Omega}$.*

Proof. Again we just show that the main arguments of the proof of [6, Theorem 4.5] continue to hold true in this Pde's setting. Assume first that ϕ is Lipschitz.

(i) The analogue of [4, Lemma 2.11] is still valid. More precisely, let u be the maximum (resp. minimum) of the solutions to $Lv = 0$ when the boundary datum ϕ satisfies the Lower (resp. Upper) (BSC) and the domain is a polyhedron Q: the existence of such a solution is established in Theorem 4.1; notice that the validity of condition (4.3) is here a consequence of the Growth Assumption (G). Then there exists a constant C depending only on the diameter of Q, $\|\phi\|_\infty$, $\|\nabla\phi\|_\infty$ such that

$$\forall\gamma \in \partial\Omega,\ \forall x \in \Omega \quad u(x) - \phi(\gamma) \leqslant C|x-\gamma|^\alpha \quad \bigl(\text{resp.}\ \phi(\gamma) - u(x) \leqslant C|x-\gamma|^\alpha\bigr).$$

Indeed the result is a consequence of the local Lipschitz continuity of u, that we established in Theorem 6.2, and of a uniform bound (as Q varies among the hypercubes whose edges have a prescribed length) of $\|\nabla u\|_{L^p(Q)}$ that follows there from the coercivity of the functional and the fact that u is a minimizer. Here such an estimate follows from the fact that, denoting again by ϕ an extension of ϕ to $\mathbb{R}^n$, from (G) we have

$$\int\limits_Q a(\nabla u)\cdot\nabla(u-\phi) = 0$$

whence

$$\alpha\int\limits_Q |\nabla u|^p\,dx \leqslant \int\limits_Q a(\nabla u)\cdot\nabla u\,dx = \int\limits_Q a(\nabla u)\cdot\nabla\phi\,dx \leqslant \int\limits_Q \bigl(\beta|\nabla u|^{p-1} + r\bigr)|\nabla\phi|\,dx.$$

(ii) Let now u be any solution to $Lv = 0$ with $v = \phi$ on $\partial\Omega$, where ϕ is Lipschitz of rank M. Following the steps of the proof of Theorem 4.5 of [6], for $\gamma \in \partial\Omega$, we consider the convex function $\phi_\gamma(x) = \phi(\gamma) + M|x-\gamma|$ and a cube Q_γ that is tangent to Ω at γ, contains Ω, and is isometric to a cube Q that does not depend on γ. Let u_γ be the maximum of the solutions on Q_γ to $Lv = 0$, $v = \phi_\gamma$ on ∂Q_γ. Since Q_γ is polyhedron we know that

$$\forall x \in Q_\gamma \quad u_\gamma(x) - \phi(\gamma) \leqslant C|x-\gamma|^\alpha \tag{6.1}$$

where C depends only on $\operatorname{diam} Q_\gamma = \operatorname{diam} Q$ and on $\|\nabla\phi_\gamma\|_\infty = M$, so not on γ itself. Since ϕ_γ is convex, then Theorem 3.1 implies that $u_\gamma \geqslant \phi_\gamma$ on Q_γ so that $u_\gamma \geqslant \phi_\gamma \geqslant \phi$ on Ω. Now u_γ is still the maximum of the solutions to $Lv = 0$ on Ω among the functions that share the same boundary datum. Again, Theorem 3.1 shows that $u_\gamma \geqslant u$ a.e. on Ω. It follows from (6.1) that

$$u(x) - \phi(\gamma) \leqslant u_\gamma(x) - \phi(\gamma) \leqslant C|x-\gamma|^\alpha \quad \text{a.e. on } \Omega.$$

Analogously one obtains that $\phi(\gamma) - u(x) \leqslant C|x-\gamma|^\alpha$ a.e. on Ω.

(iii) If u is the maximum or the minimum of the solutions to $Lv = 0$ with $v = \phi$ on $\partial\Omega$ then (a) follows from the Haar–Radò type Theorem 3.3.

(iv) Claim (b) follows as in the last lines of the proof of [5, Lemma 7] by approximating ϕ by means of Lipschitz functions and of the Comparison Principle. □

We have proved in Lemma 4.1 that if a satisfies Assumptions (A) and (G) then for every $\xi \in \mathbb{R}^n$ the level set $F_\xi = \{\eta \in \mathbb{R}^n\colon\ a(\eta) = a(\xi)\}$ is bounded by a constant depending on $|\xi|$. If this requirement is slightly strengthened the previous results hold for *every* solution to $Lv = 0$, not just for the maximum and the minimum ones.

Corollary 6.1. *Under the above assumptions assume moreover that the diameters of the level sets of a are bounded by the* (*same*) *constant. Then the conclusion of* Theorem 6.3 *does hold for every solution to* $Lv = 0$, $v = \phi$ *on* $\partial\Omega$.

Proof. It is enough to note that if u and w are solutions to $Lv = 0$ with the same boundary datum then, by Proposition 4.1, their gradients belong to the same level set. Our assumption that $w = u + \ell$ where ℓ is a Lipschitz function whose rank depends only on a: the Hölder or Lipschitz regularity of w is then inherited by that of u. □

Remark 6.2. If $p = 2$ the conclusion of Theorem 6.3 holds true when a is uniformly elliptic, i.e. there exists $\mu > 0$ such that $a(\eta) - a(\xi)) \cdot (\eta - \xi) \geqslant \mu|\eta-\xi|^2$, without assuming the Growth Assumption (G) if one assumes the existence of a solution to $-\operatorname{div} a(\nabla u) = 0$ on $\phi + W_0^{1,2}(\Omega)$. Indeed, the Growth Assumption (G) appears just at point (i) of the proof of Theorem 6.3 and ensures:

(a) the existence of a locally Lipschitz solution $u \in \phi + W_0^{1,2}(Q)$ to $-\operatorname{div} a(\nabla u) = 0$ on any hypercube Q when the boundary datum ϕ satisfies a unilateral (BSC);
(b) a uniform estimate of $\|u\|_{L^2(Q)}$ as Q varies among a family of hypercubes whose edges have a prescribed length.

If one assumes the uniform ellipticity of a instead of (G) the validity of (a) follows from [7] (see Remark 6.1). The validity of (b) follows from the fact that if u is such a solution, we have

$$0 = \int_Q a(\nabla u) \cdot (\nabla u - \nabla \phi)\, dx \geqslant \int_Q a(\nabla \phi) \cdot (\nabla u - \nabla \phi) + \mu |\nabla u - \nabla \phi|^2\, dx$$

so that, by means of Hölder's inequality,

$$\mu \|\nabla u - \nabla \phi\|_{L^2(Q)} \leqslant \big\|a(\nabla \phi)\big\|_{L^2(Q)}.$$

Remark 6.3. The conclusion of Theorem 6.3 is, from one hand, an extension of a well-known result among a-harmonic function, i.e. solutions to $-\operatorname{div} a(\nabla v) = 0$ where a is strictly monotonic and satisfies the further homogeneity assumption $a(\lambda\xi) = \lambda|\lambda|^{p-2} a(\xi)$ whenever $\lambda \in \mathbb{R}$ is non-zero [9, Theorem 6.44]. We note however that this classical result holds even when the boundary datum ϕ is Hölder, the domain is regular and a has a suitable dependence on x. The extension of the validity of our result to Hölder boundary data or regular, though non-convex domains, remains open.

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